$\sqrt{2}$ is irrational

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8:19 PM

$$\left(\frac{p}{q}\right)^2 = 2$$

$k = \min\{k : k\sqrt{2} \in \mathbb{N}\}$

$(\sqrt{2}-1)k < k$ k has same property.

$$k\frac{p}{q} \in \mathbb{N} \Rightarrow k = \text{"minimal } q\text{"}$$

$$k\left(\frac{p}{q}-1\right) = k\frac{p-q}{k} = p-q \in \mathbb{N}$$

$$\frac{p}{q} = \frac{2q-p}{p-q}$$

$$\frac{p}{q}(p-q) = \frac{p^2-qp}{q}$$

$$= p\left(\frac{p}{q}-1\right) = q\left(\frac{p^2}{q^2}-\frac{p}{q}\right)$$

$$= q\left(2-\frac{p}{q}\right) = 2q-p$$

If $\left(\frac{p}{q}\right)^2 = 2$, then also

$$\left(\frac{2q-p}{p-q}\right)^2 = 2 \text{, but}$$

$$\left(\frac{p}{q}\right) < 2 \Rightarrow p < 2q$$

$$\Rightarrow p-q < q$$

So $\frac{p}{q}$ isn't smallest

$$\Downarrow$$

$$\left(\frac{2-p/q}{p/q-1}\right)^2 = \frac{4-4p/q+2}{2-2\frac{p}{q}+1}$$

$$= \frac{6-4\frac{p}{q}}{3-3p/q} = 2$$

If $x = \frac{p}{q} = \sqrt{2}$, then same for $\frac{x+2/x}{2}$

$$\frac{x+2/x}{2} = \frac{x}{2}+\frac{1}{x} = \frac{p}{2q}+\frac{q}{p} = \frac{p^2+2q^2}{2pq}$$

Added Jan 13, proof by Rich Schwartz:

$$(\sqrt{2}+1)(\sqrt{2}-1) = 1$$

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$\overset{\shortparallel}{p/q}$ $\quad\quad$ $\overset{\shortparallel}{q/p}$ $\quad$ yet $\sqrt{2}+1$ & $\sqrt{2}-1$ have the same

denominator as they differ by

an integer.